Exercise Set #7

"Discrete Mathematics" (2025)

Exercise 6 is to be submitted on Moodle before 23:59 on April 7th, 2025

E1. Compute the values for $\mu(10!), \phi(10!)$, and $\mu(2025)$.

E2. Show that $n = \sum_{d|n} \phi(d)$ and that $\phi(n) = n \sum_{d|n} \frac{\mu(d)}{d}$.

E3. Let $\sigma(n)$ denote the sum of all positive divisors of a number n. For instance, $\sigma(6) = 1+2+3+6 = 12$. Prove that $\phi(n) + \sigma(n) \geq 2n$, for all $n \in \mathbb{N}$, and characterize all n such that equality is achieved.

E4. Let $\Lambda(n)$ be a function defined for $n \in \mathbb{Z}_{\geq 1}$ by the rule

$$\sum_{d|n} \Lambda(d) = \log n$$

Show that

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^k \text{ for some prime number } p \text{ and } k \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

E5. Let $f, g : \mathbb{N} \to \mathbb{C}$ be two functions. We define their Dirichlet product to be

$$f * g(n) = \sum_{d|n} f(d) g\left(\frac{n}{d}\right)$$

(a) Let the Dirichlet series associated with f, g be defined, respectively, as

$$F(s) = \sum_{n \geq 1} \frac{f(n)}{n^s}, G(s) = \sum_{n \geq 1} \frac{g(n)}{n^s}$$

Suppose the sums defining F and G converge absolutely for all $s > s_0$. Prove that the series

$$H(s) = \sum_{n \geq 1} \frac{f * g(n)}{n^s}$$

converges absolutely for $s > s_0$ as well, and prove that

$$H(s) = F(s)G(s)$$

(b) Let

$$\zeta(s) = \sum_{n \geq 1} n^{-s}$$

denote the Riemann Zeta function. Conclude that

$$\frac{1}{\zeta(s)} = \sum_{n \geq 1} \frac{\mu(n)}{n^s}$$

for all $s > 1$.

E6. (**Exercise to submit**)

Let $\omega(n)$ denote the number of distinct prime factors of n. Show that

$$\sum_{d|n} |\mu(d)| = 2^{\omega(n)}.$$